Logistic and Gompertz Functions

Remarks

The sigmoid curve is the s-shaped curve

Three functions of this type are the logistic growth function, the logistic decay function, and the Gompertz function

Remarks

Logistic functions are good models of biological population growth in species which have grown so large that they are near to saturating their ecosystems, or of the spread of information within societies.

They are also common in marketing, where they chart the sales of new products over time; in a different context, they can also describe the decline of demand for a product as a function of increasing price
Logistic Growth

Logistic functions combine characteristics of exponential growth and decay.

Logistic Function

\[ f(x) = \frac{c}{1 + ae^{-bx}} \]

where \(a\), \(b\), and \(c\) are constants.

\(c\) is called the carrying capacity.

If \(a > 0\), a logistic function increases when \(b > 0\) and decreases when \(b < 0\).

Example – Cell Growth

The growth of population of *Paramecium caudatum*.
Logistic Growth Model

\[ f(x) = \frac{c}{1 + ae^{-bx}} \]

If \( a > 0, \ b > 0 \)
- \( x \)-intercept – none
- \( y \)-intercept – \((0, c/(1+a))\)
- Horizontal asymptotes \( y = 0, \ y = c \)
Example

\[ P(t) = \frac{500}{1 + 7e^{-25t}} \]

t = number of days

a) What was the initial amount of bacteria?

b) When will the amount of bacteria be 300 grams?

c) What is the carrying capacity?

d) Graph the function

Example

\[ P(t) = \frac{500}{1 + 7e^{-25t}} \]

a) What was the initial amount of bacteria?

\[ P(0) = \frac{500}{1 + 7e^{-25(0)}} = \frac{500}{8} = 62.5 \text{ grams} \]

Example

b) When will the amount of bacteria be 300 grams?

\[ 300 = \frac{500}{1 + 7e^{-25t}} \]

\[ 300 = \frac{500}{1 + 7e^{-25t}} \]

\[ e^{25t} \cdot 300 = 500 \]

\[ 25t = \ln(\frac{500}{300}) \]

\[ t = \frac{\ln(\frac{500}{300})}{25} \]

\[ t = 9.4 \text{ days} \]
Example

\[ P(t) = \frac{500}{1 + 7e^{-25t}} \]

c) What is the carrying capacity?

\[ P(t) = \frac{500}{1 + 7e^{-25t}} \]

The carrying capacity is 500 grams

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Example

\[ P(t) = \frac{500}{1 + 7e^{-25t}} \]

Create logistic model

<table>
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<tr>
<th>x</th>
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<td>14</td>
<td>98</td>
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<tr>
<td>16</td>
<td>99</td>
</tr>
</tbody>
</table>
Example Create logistic model

\[
f(x) = \frac{c}{1 + ae^{-bx}} + d
\]

Logistic Logist83

Regression: logistic

Allows for vertical translation

\[
f(x) = \frac{c}{1 + ae^{-bx}}
\]
Example

Create logistic model

Regression: logist83

Equivalent to TI-83 regression

Example

A model for the number of students at Palo Alto College that have heard the latest rumor might be

\[ N(d) = \frac{P}{1 + 9e^{-0.25d}} \]

where \( P \) is the total number of students at Palo Alto and \( d \) is the number of days that have elapsed since the rumor began.

If \( P = 8000 \) students, how many students will know the latest rumor in two days? In ten days?

Example

Rumor at Palo Alto College

Two days

\[ N = \frac{8000}{1 + 9e^{-0.25 \cdot 2}} \approx 1240 \text{ students} \]

Ten days

\[ N = \frac{8000}{1 + 9e^{-0.25 \cdot 10}} \approx 4600 \text{ students} \]
Example

Rumor at Palo Alto College

Graph of first 50 days

Logistic Decay Model

\[ f(x) = \frac{c}{1 + ae^{-bx}} \]

If \( a > 0 \), \( b < 0 \)

- x-intercept: none
- y-intercept: \((0, c/(1+a))\)
- Horizontal asymptotes \( y = 0 \), \( y = c \)

Logistic Growth Model

\[ f(x) = \frac{c}{1 + ae^{-bx}} \]

\( y \) intercept: \( y_0 = \frac{c}{1 + a} \)

\( b < 0 \)
Gompertz Function

\[ f(x) = C e^{R e^{at}} \]

- \( 0 < R < 1 \) is the expected rate of growth of a population
- \( a \) is the proportion of the initial population
- \( C \) is the carrying capacity

In Biology - The Gompertz growth law has been shown to provide a good fit for the growth data of numerous tumors

Example

The number of micro DVD players sold by the Palo Alto Manufacturing Company each month is given by \( N = 10,000(0.2)^{0.3t} \)

(a) How many DVD players will be sold the first month?
(b) Graph the function for the first year
(c) What is the predicted upper limit on sales?

\[
N(1) = 10,000(0.2)^{0.3} = 6,170 \text{ units}
\]
The number of micro DVD players sold by the Palo Alto Manufacturing Company each month is given by $N = 10,000(0.2)^t$.

(b) Graph the function for the first year.

To graph a Gompertz function on a TI-83/84 we must use Y1 and Y2.

To graph Y1: $Y1 = 10,000(0.2)^x$, where $x$ is the time in months.

To graph Y2: $Y2 = 10,000(0.2)^{2x}$ for the second year.

(c) What is the predicted upper limit on sales?

Upper limit = 10,000 units.