Higher-Degree Polynomial Functions

Polynomials

A polynomial is an expression that is constructed from one or more variables and constants, using only the operations of addition, subtraction, multiplication, and constant positive whole number exponents.

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0 \]

Is the standard form of a polynomial where \( n \) is a non-negative integer and \( a_n \) is called the leading coefficient.

For example, \( 2x^2 + 4x - 3 \) is a polynomial

\( 2x^2 + 4x - 3x^{3/2} \) is not a polynomial because it involves division by a variable and also because it has an exponent that is not a positive whole number.

Polynomials

<table>
<thead>
<tr>
<th>Degree</th>
<th>Name of Polynomial Function</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>Linear</td>
<td>( f(x) = 2x + 5 )</td>
</tr>
<tr>
<td>Second</td>
<td>Quadratic</td>
<td>( f(x) = 3x^2 - 5x + 2 )</td>
</tr>
<tr>
<td>Third</td>
<td>Cubic</td>
<td>( f(x) = x^3 - 2x - 1 )</td>
</tr>
<tr>
<td>Fourth</td>
<td>Quartic</td>
<td>( f(x) = x^4 - 3x^3 + 7x - 6 )</td>
</tr>
<tr>
<td>Fifth</td>
<td>Quintic</td>
<td>( f(x) = 2x^5 + 3x^4 - x^3 + x^2 )</td>
</tr>
</tbody>
</table>

Domain: all real numbers. There are no holes or breaks in the graph.
End Behavior

We want to examine what happens to the output (y-values) when the input (x-values) to a polynomial are very large.

We will do this by examining two functions:

\[ y = 2x^3 + 4x - 8 \quad \text{and} \quad y = 2x^3 \]

with increasing x-values.

Explore

<table>
<thead>
<tr>
<th>x</th>
<th>V1</th>
<th>V2</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>298</td>
<td>289</td>
</tr>
<tr>
<td>V3</td>
<td>250000000</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>V1</th>
<th>V2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>189</td>
<td>186</td>
</tr>
<tr>
<td>V1</td>
<td>200000000</td>
<td></td>
</tr>
</tbody>
</table>

End Behavior

Polynomials tend to the value of the leading term for large inputs (x-values):

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

As \( x \to \infty \) we have \( f(x) \to a_n x^n \).

The engine determines end behavior.
End Behavior

- $n \geq 2$ and even, $a_n > 0$
- $n \geq 3$ and odd, $a_n > 0$

---

End Behavior

- $n \geq 2$ and even, $a_n < 0$
- $n \geq 3$ and odd, $a_n < 0$

---

Example

Determine end behavior

$-3.4x^4 - 4.2x^3 + 6.1x^2 + 9.7x - 5.3$
Example

Determine end behavior

\[-3.4x^4 - 4.2x^3 + 6.1x^2 + 9.7x - 5.3\]

Remarks

In general, a polynomial of degree n

+ Has at most n x-intercepts
+ Has at most n-1 turning points

A polynomial of even degree has both ends opening up, or has both ends opening down

A polynomial of odd degree has one end opening up and the other end opening down and has at least one x-intercept

Quadratic

2 x-intercepts

e.g. \( y = (x - 3)(x + 5) \)

1 x-intercept

e.g. \( y = (x - 2)^2 \)

No real x-intercept

e.g. \( y = (x + 1)^2 + 2 \)

Has 1 turning points
Cubic

3 x-intercepts
e.g. $y = (x+5)(x+1)(x-3)$

1 x-intercept
e.g. $y = (x+4)(x^2+2)$

Has 2 turning points

Explore

2 x-intercepts
One turn

4 x-intercepts
Three turns

6 x-intercepts
Five turns

Remarks

Turning points are extremely important as they are local extrema points of the graph of a function.

In business we wish to maximize revenue, minimize cost, and even more important, we really want to maximize our profit.

The points are referred to as local maximum or local minimum points. We also refer to them as relative maximum or relative minimum points.

The local maxima will look like the top of a hill and the local minima will look like the bottom of a valley.
Turning Point

The graph of a function turns at a local maximum or minimum of the function

Turning point and local maxima

Turning point and local minima

Remarks

When a graph has a highest point, we refer to this point as the absolute maximum or the absolute minimum

When a graph has a lowest point, we refer to this point as the absolute minimum or the absolute minimum

When answering questions about extrema, read the question carefully to determine if you are being asked for the x-value or the y-value.

For example if we graph cost as a function of the number of units produced, the minimum cost is a y-value and the number of units required for the minimum cost is a x-value

Example

Find the maximum revenue of CD players

\[ R(x) = -0.1x^3 + 15x^2 - 25x + 11 \]

We use the Y= editor

[Diagram of Y= editor with equation input and graph display]
Example

Find the maximum revenue of CD players

\[ R(x) = -1x^3 + 15x^2 - 25x + 11 \]

We use an appropriate window

---

Example

Find the maximum revenue of CD players

\[ R(x) = -1x^3 + 15x^2 - 25x + 11 \]

We find the x value

---

Example

Find the maximum revenue of CD players

\[ R(x) = -1x^3 + 15x^2 - 25x + 11 \]

We find the x value

---

Example

Find the maximum revenue of CD players

\[ R(x) = -1x^3 + 15x^2 - 25x + 11 \]

We find the x value

---
Example

Find the maximum revenue of CD players
\[ R(x) = -1.1x^3 + 15x^2 - 25x + 11 \]
We want to find \( R(99) \)

We can only sell whole CD players

Example

Find the maximum revenue of CD players
\[ R(x) = -1.1x^3 + 15x^2 - 25x + 11 \]

We should sell 99 CD players to obtain the maximum revenue of $47,521.10

Increasing

A function \( f \) is increasing on an interval \( I \) if, for any choice of \( x_1 \) and \( x_2 \) in \( I \), with \( x_1 < x_2 \) we have \( f(x_1) < f(x_2) \)

When finding intervals where a function is increasing we are finding the \( x \)-values
Decreasing

A function \( f \) is **decreasing** on an interval \( I \) if, for any choice of \( x_1 \) and \( x_2 \) in \( I \), with \( x_1 < x_2 \) we have \( f(x_1) > f(x_2) \).

When finding intervals where a function is decreasing we are finding the \( x \)-values.

Remarks

To find intervals where a function is increasing or decreasing, we first must identify the turning points (the local extrema).

Interval Increasing

Example

Find local extrema, and intervals where the function is increasing or decreasing

\[ f(x) = 2x^3 + 3x^2 - 72x + 12 \]

Enter function into \( Y= \) editor
Example

Find local extrema, and intervals where the function is increasing or decreasing

\[ f(x) = 2x^3 + 3x^2 - 72x + 12 \]

Use a suitable window

```
WINDOW
xmin = -10
xmax = 10
xscale = 1.
ymin = -200.
ymax = 250.
yscale = 1.
xres = 2.
```

Example

Find local extrema, and intervals where the function is increasing or decreasing

\[ f(x) = 2x^3 + 3x^2 - 72x + 12 \]

Find the local minimum

Minimum

\[ x_c = 3, \quad y_c = 123. \]

Example

Find local extrema, and intervals where the function is increasing or decreasing

\[ f(x) = 2x^3 + 3x^2 - 72x + 12 \]

Find the local maximum

Maximum

\[ x_c = 4, \quad y_c = 220. \]
Example

Find local extrema, and intervals where the function is increasing or decreasing

\[ f(x) = 2x^3 + 3x^2 - 72x + 12 \]

We have a local minimum of \( y = -123 \) at \( x = 3 \)
We have a local maximum of \( y = 220 \) at \( x = -4 \)
The graph is increasing from \( x = -\infty \) to \( x = -4 \)
then decreasing from \( x = -4 \) to \( x = 3 \)
then increasing from \( x = 3 \) to \( \infty \)

In interval notation the graph is increasing \( (-\infty, -4) \cup (3, \infty) \)
and decreasing \( (-4, 3) \)