Characterization of Invertible Matrices

Remarks
For an invertible matrix $A$,

$$A^{-1}Ax = x \text{ for all } x \in \mathbb{R}^n$$

and

$$AA^{-1}x = x \text{ for all } x \in \mathbb{R}^n$$

Theorem 8 - Invertible Matrix Theorem
Let $A$ be a square $n \times n$ matrix. The following statements are equivalent. i.e. for a given matrix $A$, they are all either true or false

- $A$ is an invertible matrix
- $A$ is row equivalent to $I_n$
- $A$ has $n$ pivot positions
- The equation $Ax = \mathbf{0}$ has only the trivial solution
- The columns of $A$ are linearly independent
- The linear transformation $x \rightarrow Ax$ is one-to-one
- The equation $Ax = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^n$
- The columns of $A$ span $\mathbb{R}^n$
- The linear transformation $x \rightarrow Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- There is a $n \times n$ matrix $C$ such that $AC = I_n$
- There is a $n \times n$ matrix $D$ such that $DA = I_n$
- $A^T$ is an invertible matrix
Example

Determine if invertible

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
-4 & 11 & 1 \\
2 & 7 & 3
\end{bmatrix}
\]

We see each row has a pivot, hence by the IMT, matrix \( A \) is invertible.

Example

Determine if invertible

\[
A = \begin{bmatrix}
2 & 2 & 7 \\
1 & 1 & -2 \\
-3 & -3 & 5
\end{bmatrix}
\]

If \( A \) has two identical columns, then its columns are not linearly independent.

By the IMT, \( A \) cannot be invertible.

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Similar list from Gil Strang's Book

<table>
<thead>
<tr>
<th>Nonsingular</th>
<th>Singular</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A ) is invertible</td>
<td>( A ) is not invertible</td>
</tr>
<tr>
<td>The columns are independent</td>
<td>The columns are dependent</td>
</tr>
<tr>
<td>The rows are independent</td>
<td>The rows are dependent</td>
</tr>
<tr>
<td>The determinant is not zero</td>
<td>The determinant is zero</td>
</tr>
<tr>
<td>( Ax = 0 ) has one solution ( x = 0 )</td>
<td>( Ax = 0 ) has infinitely many solutions</td>
</tr>
<tr>
<td>( Ax = b ) has one solution ( x = A^{-1}b )</td>
<td>( Ax = b ) has no solution or infinitely many</td>
</tr>
<tr>
<td>( A ) has ( n ) (nonzero) pivots</td>
<td>( A ) has ( r &lt; n ) pivots</td>
</tr>
<tr>
<td>( A ) has full rank ( r = n )</td>
<td>( A ) has rank ( r &lt; n )</td>
</tr>
<tr>
<td>The reduced row echelon form is ( R = I )</td>
<td>The reduced row echelon form is ( R ) has at least one zero row</td>
</tr>
<tr>
<td>The column space is all of ( \mathbb{R}^n )</td>
<td>The column space has dimension ( r &lt; n )</td>
</tr>
<tr>
<td>The row space is all of ( \mathbb{R}^m )</td>
<td>The row space has dimension ( r &lt; n )</td>
</tr>
<tr>
<td>All eigenvalues are nonzero</td>
<td>Zero is an eigenvalue of ( A )</td>
</tr>
<tr>
<td>( A^T A ) is symmetric positive definite</td>
<td>( A^T A ) is only semidefinite</td>
</tr>
<tr>
<td>( A ) has ( n ) (positive) singular values</td>
<td>( A ) has ( r &lt; n ) singular values</td>
</tr>
</tbody>
</table>
**Application - Migration**
The migration matrix model is used to predict population movement using past performance.

Suppose we have two population centers.

Suppose 5% of Westsiders migrate to Eastside each year and 10% of Eastsiders migrate to Westside each year.

Suppose the Eastside population is 120,000, Westside population is 80,000 and that we want to predict the populations after 1, 5, and 10 years.

We are given

<table>
<thead>
<tr>
<th>Year 0</th>
<th>Every year</th>
</tr>
</thead>
<tbody>
<tr>
<td>E=120,000</td>
<td>0.10E ⇒ W, and 0.90E remains E</td>
</tr>
<tr>
<td>W=80,000</td>
<td>0.05W ⇒ E, and 0.95W remains W</td>
</tr>
</tbody>
</table>

then each year

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90E + 0.05W ⇒ E</td>
<td>0.95W + 0.10E ⇒ W</td>
</tr>
</tbody>
</table>

Our recursive formulas

\[
E_i = 0.90E_{i-1} + 0.05W_{i-1} \\
W_i = 0.10E_{i-1} + 0.95W_{i-1}
\]

Our beginning population

\[
x_0 = \begin{bmatrix} 120,000 \\ 80,000 \end{bmatrix}
\]
A simple way to build the migration matrix

$$ M = \begin{bmatrix} .90 & .05 \\ .10 & .95 \end{bmatrix} $$

Observe the sum of each column is 1 corresponding to 100% probability

After 1 year

The total population remains at 200,000

After 2 years

We have two ways of calculating this
After 5 years

\[
\begin{bmatrix}
90331.1 \\
109669.1
\end{bmatrix}
\]

After 10 years

\[
\begin{bmatrix}
77166.6 \\
122833.1
\end{bmatrix}
\]

Suppose we want to build a factory in one of these population centers and want to predict the population after many years.

To predict long range (end behavior) we find \( \lim_{k \to \infty} M_k \).

\[
\lim_{k \to \infty} M_k = \lim_{k \to \infty} \begin{bmatrix}
.90 & .05 \\
.10 & .95
\end{bmatrix}
\]

We can examine \( M^1, M^2, M^3, \ldots \) to help predict the end behavior.

\[
M^1 = \begin{bmatrix}
9 & .05 \\
1 & .95
\end{bmatrix}, \quad M^2 = \begin{bmatrix}
.815 & .093 \\
.185 & .908
\end{bmatrix}, \quad M^3 = \begin{bmatrix}
.743 & .129 \\
.258 & .871
\end{bmatrix}
\]

\[
M^4 = \begin{bmatrix}
.681 & .159 \\
.318 & .840
\end{bmatrix}, \quad M^5 = \begin{bmatrix}
.629 & .185 \\
.371 & .815
\end{bmatrix}, \quad M^6 = \begin{bmatrix}
.585 & .208 \\
.415 & .729
\end{bmatrix}
\]
Using TI to Predict End Behavior

We predict that \( \lim_{k \to \infty} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \)
and \( \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} [120,000] = \begin{bmatrix} 66,667 \\ 133,333 \end{bmatrix} \)
If this is true, we might suspect \( \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} 66,667 \\ 133,333 \end{bmatrix} = \begin{bmatrix} 66,667 \\ 133,333 \end{bmatrix} \)

Remarks

Another way to predict end behavior we observe when \( x \) is at steady state \( Mx = x \)
\( \Rightarrow Mx =Ix \Rightarrow Mx -Ix =0 \Rightarrow (M-I)x =0 \)

\[ M-I = \begin{bmatrix} .90 & .05 \\ .10 & .95 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.10 & .05 \\ .10 & -.05 \end{bmatrix} \]

Our augmented matrix
\( \begin{bmatrix} -.10 & .05 & 0 \\ .10 & -.05 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E - \frac{1}{2} W = 0 \)
Our initial population is 120,000 + 80,000
gets us a second equation of \( E + W = 200,000 \)
Together we have
\[
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
1 & 1 & 200,000 \\
1 & 1 & 33,333
\end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\
1 & 1 & 133,333 \\
1 & 1 & 33,333
\end{bmatrix}
\]

If the combined population increased each year, by for example 8%, we can easily modify our model

\[
M = \begin{bmatrix} .90 & .05 \\
.10 & .95 \end{bmatrix} \text{ and } x_n = 1.08^{n} \begin{bmatrix} .90 & .05 \\
.10 & .95 \end{bmatrix} x_{n-1}
\]

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Example

Rats in an enclosed space

Suppose each cycle
- 12% of \( P \Rightarrow Q \)
- 6% of \( Q \Rightarrow P \)
- 8% of \( Q \Rightarrow R \)
- 10% of \( R \Rightarrow Q \)

Suppose initially \( P = 100, Q = 80, R = 90 \)

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Example

Rats in an enclosed space

Our equations
\[
P_2 = 0.88P_1 + 0.06Q_1 \\
Q_2 = 0.12P_1 + 0.86Q_1 + 0.10R_1 \\
R_2 = 0.08Q_1 + 0.90R_1
\]

Our migration matrix
\[
M = \begin{bmatrix} .88 & .06 & 0 \\
.12 & .86 & .10 \\
0 & .08 & .90 \end{bmatrix}
\]
During each 10-year period, 8% of the people in Alamo Heights move to the surrounding area, and the rest remain in Alamo Heights. Also 3% of the surrounding area move into Alamo Heights.

In 2005 there were 100,000 residents in Alamo Heights and 200,000 in the surrounding area. Set up the migration matrix and compute the population vector for 2015 (one ten-year period).

\[
\begin{pmatrix}
0.92 & 0.03 \\
0.08 & 0.97 \\
\end{pmatrix} \begin{pmatrix}
100,000 \\
200,000 \\
\end{pmatrix}
= \begin{pmatrix}
98,000 \\
202,000 \\
\end{pmatrix}
\]

Invertible Transformation

A linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) is said to be invertible if there exists a function \( S : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
S(T(x)) = x \text{ for all } x \in \mathbb{R}^n
\]

and

\[
T(S(x)) = x \text{ for all } x \in \mathbb{R}^n
\]
Theorem 9

Let $T: R^n \rightarrow R^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix. In that case, the linear transformation $S$ given by $S(x) = A^{-1}x$ is the unique function satisfying

$S(T(x)) = x$ for all $x \in R^n$

and

$T(S(x)) = x$ for all $x \in R^n$

Example

$T(x_1, x_2) = (7x_1 + 9x_2, 4x_1 + 5x_2)$

$T(x)$ is a transformation from $R^2$ to $R^2$

The matrix associated with the transformation $A = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$

We see that $A x = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 + 9x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$

The inverse of the transformation matrix $A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$

and $A^{-1} x = \begin{bmatrix} -5 \\ 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{bmatrix}$

Example

$T^{-1}(x_1, x_2)$ can be found by $A(A^{-1} x) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7(-5x_1 + 9x_2) + 9(4x_1 - 7x_2) \\ 4(-5x_1 + 9x_2) + 5(4x_1 - 7x_2) \end{bmatrix}$

$= \begin{bmatrix} -35x_1 + 63x_2 + 36x_1 - 63x_2 \\ -20x_1 + 36x_2 + 20x_1 - 35x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So, $T^{-1}(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$